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Physical Algebras in Four Dimensions. I. The Clifford Algebra in Minkowski Spacetime

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This paper provides a compact, unified framework for the description of physical fields in spacetime. We combine features of the traditional vector, matrix, tensor, spinor, quaternion, and dyadic methods into a simple easy-to-use scheme. The field description is both matrix free and coordinate free. This construction is achieved by using the differential forms of Minkowski spacetime to realize a Clifford algebra of dimension 16. We should note that this algebra is distinct from either the algebra of Dirac matrices, or the algebra of the Majorana matrices. A novel characteristic of the algebraic structure is that an inverse, and consequently division by a vector or an antisymmetric tensor field of any rank, is perfectly well defined. Products and inverses of all antisymmetric tensor fields in spacetime are worked out in terms of the usual vector notation. Among useful features of this scheme is the description of duals as products by a basis element. Moreover, this field description is intrinsically Lorentz covariant, and the algebraic product preserves the covariance. Many examples are given in order to illustrate the practical value of the formalism presented herein.

1. INTRODUCTION

Historically, there have been several schemes introduced specifically for the description of physics in Minkowski spacetime. Among algebraic schemes, the algebra of Dirac matrices [1] and the theory of spinors [2–7] have been instrumental in successfully describing relativistic fields. It is known that these two formalisms are intimately related to the theory of Clifford algebras [6–12]. A mathematical scheme which has a primarily geometrical interpretation is the exterior algebra of Grassmann and Cartan, which has been incorporated into the theory of differential forms [13–15].

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This appears to provide a most natural setting for the description of physical tensor fields, as for example in electromagnetism and relativity [14–16]. A more recent formalism which has both geometric and algebraic aspects is the theory of twistors of Penrose [17–19].

In this paper, we discuss the geometrical properties of the Clifford algebra in Minkowski spacetime, realized in terms of differential forms [20–24]. This realization combines the useful features from the theory of differential forms such as geometrical tensor description and duality, with the manipulative ease of Clifford algebras. Central to this scheme is an associative product between differential forms which combines elements of the usual quaternion, exterior (Grassmann), spinor, and matrix products. The elements of the algebra are antisymmetric tensor fields in spacetime—which are the objects of direct physical relevance. An important and novel property of the construction is that the inverse of each field, and consequently division by a field, is perfectly well defined. Even though many of the geometrical and algebraic aspects discussed are separately known, it is the joining of the theory of differential forms to the theory of Clifford algebras that possesses unique calculational and conceptual advantages over more familiar formalisms. Among these, we must stress the complete absence of representation matrices in an intrinsic coordinate-free description of tensor fields.

It is important to note that the algebra discussed here is *not* isomorphic to either the algebra of the Dirac matrices, or the algebra of the Majorana matrices. The algebra of Dirac matrices corresponds to a Clifford algebra that is twice as large, and the relationship may be thought of as a complexification. However, the real part of the algebra of Dirac matrices is the algebra of the Majorana matrices, which is of the same dimension as the algebra discussed here, but is a distinct algebra. This important point has been extensively discussed in [24]. (The real Clifford algebra in Minkowski spacetime is elsewhere denoted either as N_4, $A^{1,3}$, or in terms of its matrix representation space as $H(2)$; see [24].)

This particular algebraic scheme has been previously used in the description of physics in spacetime [20, 22, 24]. Here, we present in detail for the first time the actual machinery for practical manipulations. This paper satisfies the need for a systematic method of manipulating general classes of tensor fields in this algebra, as well as for a symbolism that lends itself to formal, as well as geometrically intuitive proofs. We have utilized as much as possible familiar notation such as the ordinary vector algebra, so that the formalism can be appreciated and used by the nonspecialist.

Our exposition begins with a brief review of differential forms and duality that fixes the notation (Section 2). The product in the algebra is a multiplication between differential forms, called the "vee product," whose definition and properties are discussed in Section 3. In Section 4, the vee product rules are established for forms in Minkowski spacetime, and the manipulatory consistency is illustrated by several examples.

One of the key points of the construction is the recognition that the vee product can be used to systematically find duals of fields that are encountered in physics. The "duality theorem" discussed in Section 5 reduces the duality from an often involved index operation to a simple algebraic product. Also in this section, we show how the differential forms in Minkowski spacetime define a finite group under the vee product. The vee group multiplication table is given in Table 1.

Section 6 establishes the contact with physics by defining objects called "tensor types." These are antisymmetric tensor fields expanded on the differential form basis, which retain all the usual properties from the theory of differential forms, yet gain additional features peculiar and unique to the Clifford algebraic basis (Section 7). These features enable us to obtain rules for the products and inverses of combinations of tensor fields. We are in addition able to utilize the familiar vector algebra in three dimensions to describe all the vee products of fields in Minkowski spacetime, which are listed in Table 2.

Lastly, we demonstrate that even though this algebra is neither a normed nor a division algebra, it possesses certain useful features of both, illustrating the Frobenius–Hurwitz theorems and their extension (Section 8).

Related, but distinct, discussions of Clifford algebras in four dimensions include references [25–36]. There are several points in common between these, our own work, and the more familiar treatment of Clifford algebras in terms of the Dirac gamma matrices. However, in addition to the basic differences discussed in the body of this paper, note the frequent use of a metric different from the Minkowski metric by other authors; the absence of a direct product between higher-rank fields, or, when such products are defined, their inequivalence to ours; and the use of complex instead of real field components. We believe that the recent proliferation of studies on Clifford algebras is an indication of the growing interest in a more intrinsic description of physical fields.

2. Differential Forms and the Geometry of the Minkowski Space

In this section we review those parts of the formalism of differential forms that are necessary for our subsequent discussion. In essence, the differential form basis describes the geometry of Minkowski spacetime in a direct and intrinsic manner. (The discussion here concentrates on the intuitive and geometrical aspects; a more rigorous treatment may be found in references [13–15].)

TABLE 1
Vee Group Multiplication Table of Basis Forms in Minkowski Spacetime

$\vee$	σ^1	σ^2	σ^3	$*\sigma^1$	$*\sigma^2$	$*\sigma^3$	η
σ^1	-1	$*\sigma^3$	$-*\sigma^2$	η	σ^3	$-\sigma^2$	$-*\sigma^1$
σ^2	$-*\sigma^3$	-1	$*\sigma^1$	$-\sigma^3$	η	σ^1	$-*\sigma^2$
σ^3	$*\sigma^2$	$-*\sigma^1$	-1	σ^2	$-\sigma^1$	η	$-*\sigma^3$
$*\sigma^1$	η	σ^3	$-\sigma^2$	-1	$*\sigma^3$	$-*\sigma^2$	$-\sigma^1$
$*\sigma^2$	$-\sigma^3$	η	σ^1	$-*\sigma^3$	-1	$*\sigma^1$	$-\sigma^2$
$*\sigma^3$	σ^2	$-\sigma^1$	η	$*\sigma^2$	$-*\sigma^1$	-1	$-\sigma^3$
η	$-*\sigma^1$	$-*\sigma^2$	$-*\sigma^3$	$-\sigma^1$	$-\sigma^2$	$-\sigma^3$	1
σ^4	$-\sigma^1 \wedge \sigma^4$	$-\sigma^2 \wedge \sigma^4$	$-\sigma^3 \wedge \sigma^4$	$\bar{\sigma}^1$	$\bar{\sigma}^2$	$\bar{\sigma}^3$	$-\omega$
$\sigma^1 \wedge \sigma^4$	σ^4	$-\bar{\sigma}^3$	$\bar{\sigma}^2$	ω	$\sigma^3 \wedge \sigma^4$	$-\sigma^2 \wedge \sigma^4$	$\bar{\sigma}^1$
$\sigma^2 \wedge \sigma^4$	$\bar{\sigma}^3$	σ^4	$-\bar{\sigma}^1$	$-\sigma^3 \wedge \sigma^4$	ω	$\sigma^1 \wedge \sigma^4$	$\bar{\sigma}^2$
$\sigma^3 \wedge \sigma^4$	$-\bar{\sigma}^2$	$\bar{\sigma}^1$	σ^4	$\sigma^2 \wedge \sigma^4$	$-\sigma^1 \wedge \sigma^4$	ω	$\bar{\sigma}^3$
$\bar{\sigma}^1$	$-\omega$	$-\sigma^3 \wedge \sigma^4$	$\sigma^2 \wedge \sigma^4$	$-\sigma^4$	$\bar{\sigma}^3$	$-\bar{\sigma}^2$	$\sigma^1 \wedge \sigma^4$
$\bar{\sigma}^2$	$\sigma^3 \wedge \sigma^4$	$-\omega$	$-\sigma^1 \wedge \sigma^4$	$-\bar{\sigma}^3$	$-\sigma^4$	$\bar{\sigma}^1$	$\sigma^2 \wedge \sigma^4$
$\bar{\sigma}^3$	$-\sigma^2 \wedge \sigma^4$	$\sigma^1 \wedge \sigma^4$	$-\omega$	$\bar{\sigma}^2$	$-\bar{\sigma}^1$	$-\sigma^4$	$\sigma^3 \wedge \sigma^4$
ω	$\bar{\sigma}^1$	$\bar{\sigma}^2$	$\bar{\sigma}^3$	$-\sigma^1 \wedge \sigma^4$	$-\sigma^2 \wedge \sigma^4$	$-\sigma^3 \wedge \sigma^4$	$-\sigma^4$

The Minkowski space is four-dimensional, and is described by the coordinates $x^1 = x$, $x^2 = y$, $x^3 = z$, $x^4 = t$. It is convenient to use Greek indices $\mu = 1,2,3,4$ to denote the coordinates as x^μ. The coordinate axes are assumed to be (and otherwise can always be made) orthogonal. The exterior derivative acting on each coordinate function x^μ is simply the differential dx^μ. This differential is a directed quantity, which points in the positive direction of the x^μ axis. The four differentials $\{dx^1, dx^2, dx^3, dx^4\}$ therefore define an orthonormal frame or tetrad in the space. They can, and will be, used in the manner of unit vectors in the Minkowski space. (Note that in this treatment, we do not identify the tangent bundle with the partial derivatives, since the space is flat.)

Now define the Grassmann–Cartan exterior (or wedge) product $\wedge$ by the following formal definition:

$$\left.\begin{aligned} dx^\mu \wedge dx^\nu &= -dx^\nu \wedge dx^\mu, \qquad \mu \neq \nu \\ dx^\mu \wedge dx^\mu &= 0 \end{aligned}\right\} \quad \text{antisymmetry,} \tag{1a, 1b}$$

$$\left.\begin{aligned} dx^\mu \wedge (dx^\nu \wedge dx^\lambda) &= (dx^\mu \wedge dx^\nu) \wedge dx^\lambda \\ &= dx^\mu \wedge dx^\nu \wedge dx^\lambda \end{aligned}\right\} \quad \text{associativity,} \tag{1c}$$

TABLE 1
Continued

σ^4	$\sigma^1 \wedge \sigma^4$	$\sigma^2 \wedge \sigma^4$	$\sigma^3 \wedge \sigma^4$	$\bar{\sigma}^1$	$\bar{\sigma}^2$	$\bar{\sigma}^3$	ω
$\sigma^1 \wedge \sigma^4$	$-\sigma^4$	$\bar{\sigma}^3$	$-\bar{\sigma}^2$	ω	$\sigma^3 \wedge \sigma^4$	$-\sigma^2 \wedge \sigma^4$	$-\bar{\sigma}^1$
$\sigma^2 \wedge \sigma^4$	$-\bar{\sigma}^3$	$-\sigma^4$	$\bar{\sigma}^1$	$-\sigma^3 \wedge \sigma^4$	ω	$\sigma^1 \wedge \sigma^4$	$-\bar{\sigma}^2$
$\sigma^3 \wedge \sigma^4$	$\bar{\sigma}^2$	$-\bar{\sigma}^1$	$-\sigma^4$	$\sigma^2 \wedge \sigma^4$	$-\sigma^1 \wedge \sigma^4$	ω	$-\bar{\sigma}^3$
$\bar{\sigma}^1$	ω	$\sigma^3 \wedge \sigma^4$	$-\sigma^2 \wedge \sigma^4$	$-\sigma^4$	$\bar{\sigma}^3$	$-\bar{\sigma}^2$	$-\sigma^1 \wedge \sigma^4$
$\bar{\sigma}^2$	$-\sigma^3 \wedge \sigma^4$	ω	$\sigma^1 \wedge \sigma^4$	$-\bar{\sigma}^3$	$-\sigma^4$	$\bar{\sigma}^1$	$-\sigma^2 \wedge \sigma^4$
$\bar{\sigma}^3$	$\sigma^2 \wedge \sigma^4$	$-\sigma^1 \wedge \sigma^4$	ω	$\bar{\sigma}^2$	$-\bar{\sigma}^1$	$-\sigma^4$	$-\sigma^3 \wedge \sigma^4$
ω	$-\bar{\sigma}^1$	$-\bar{\sigma}^2$	$-\bar{\sigma}^3$	$-\sigma^1 \wedge \sigma^4$	$-\sigma^2 \wedge \sigma^4$	$-\sigma^3 \wedge \sigma^4$	σ^4
1	$-\sigma^1$	$-\sigma^2$	$-\sigma^3$	$\ast\sigma^1$	$\ast\sigma^2$	$\ast\sigma^3$	$-\eta$
σ^1	1	$-\ast\sigma^3$	$\ast\sigma^2$	η	σ^3	$-\sigma^2$	$\ast\sigma^1$
σ^2	$\ast\sigma^3$	1	$-\ast\sigma^1$	$-\sigma^3$	η	σ^1	$\ast\sigma^2$
σ^3	$-\ast\sigma^2$	$\ast\sigma^1$	1	σ^2	$-\sigma^1$	η	$\ast\sigma^3$
$\ast\sigma^1$	$-\eta$	$-\sigma^3$	σ^2	-1	$\ast\sigma^3$	$-\ast\sigma^2$	σ^1
$\ast\sigma^2$	σ^3	$-\eta$	$-\sigma^1$	$-\ast\sigma^3$	-1	$\ast\sigma^1$	σ^2
$\ast\sigma^3$	$-\sigma^2$	σ^1	$-\eta$	$\ast\sigma^2$	$-\ast\sigma^1$	-1	σ^3
η	$\ast\sigma^1$	$\ast\sigma^2$	$\ast\sigma^3$	$-\sigma^1$	$-\sigma^2$	$-\sigma^3$	-1

$$dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p} = (-1)^{\pi}\, dx^{\lambda_1} \wedge \cdots \wedge dx^{\lambda_p}, \qquad \mu_1 \neq \cdots \neq \mu_p$$

permutation symmetry, (1d)

$$dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p} = 0 \qquad \text{if any pair of indices is equal.} \qquad (1e)$$

Here, $(-1)^{\pi}$ is the sign of the permutation that rearranges the indices.

$$\pi = \begin{pmatrix} \mu_1 \cdots \mu_p \\ \lambda_1 \cdots \lambda_p \end{pmatrix}. \qquad (2)$$

Condition (1d) means that we can interchange any two indices, and the result is a change in sign. For example,

$$dx^1 \wedge dx^3 \wedge dx^4 = -dx^4 \wedge dx^3 \wedge dx^1 = -dx^3 \wedge dx^1 \wedge dx^4, \qquad \text{etc.} \qquad (3)$$

The object in (3) is called a 3-form; in general, the exterior product of p differentials as in (1d) is called a p-form, or a form of rank p. Because of the permutation symmetry, we can display a one-to-one correspondence between these basis forms and the permutation group. Specifically, in a

TABLE 2
The General Vee Product of Fields in Minkowski Spacetime

$$\begin{pmatrix} 0 & a & \mathbf{c} \vee \sigma^4 \\ 0 & \omega \vee b & \eta \vee \mathbf{d} \end{pmatrix} \vee \begin{pmatrix} 0 & e & \mathbf{g} \vee \sigma^4 \\ 0 & \omega \vee f & \eta \vee \mathbf{h} \end{pmatrix} =$$

Rank			
(1×1)	$\Bigg(\begin{matrix} (a_\mu e^\mu) \\ 0 \end{matrix}$	$\begin{matrix} 0 \\ 0 \end{matrix}$	$\begin{matrix} (e^4 \mathbf{a} - a^4 \mathbf{e}) \vee \sigma^4 \\ -\eta \vee (\mathbf{a} \times \mathbf{e}) \end{matrix} \Bigg)$
$+(1 \times 2)$	$\Bigg(\begin{matrix} 0 \\ 0 \end{matrix}$	$\begin{matrix} -\mathbf{a} \times \mathbf{h} - a^4 \mathbf{g} - (\mathbf{a} \cdot \mathbf{g}) \sigma^4 \\ \omega \vee \left(\mathbf{a} \times \mathbf{g} - a^4 \mathbf{h} - (\mathbf{a} \cdot \mathbf{h}) \sigma^4 \right) \end{matrix}$	$\begin{matrix} 0 \\ 0 \end{matrix} \Bigg)$
$+(2 \times 1)$	$\Bigg(\begin{matrix} 0 \\ 0 \end{matrix}$	$\begin{matrix} -\mathbf{d} \times \mathbf{e} + e^4 \mathbf{c} + (\mathbf{c} \cdot \mathbf{e}) \sigma^4 \\ \omega \vee \left(-\mathbf{c} \times \mathbf{e} - e^4 \mathbf{d} - (\mathbf{d} \cdot \mathbf{e}) \sigma^4 \right) \end{matrix}$	$\begin{matrix} 0 \\ 0 \end{matrix} \Bigg)$
$+(2 \times 2)$	$\Bigg(\begin{matrix} (\mathbf{c} \cdot \mathbf{g}) - (\mathbf{d} \cdot \mathbf{h}) \\ (-(\mathbf{c} \cdot \mathbf{h}) - (\mathbf{d} \cdot \mathbf{g}))\, \omega \end{matrix}$	$\begin{matrix} 0 \\ 0 \end{matrix}$	$\begin{matrix} (-\mathbf{c} \times \mathbf{h} - \mathbf{d} \times \mathbf{g}) \vee \sigma^4 \\ \eta \vee (\mathbf{c} \times \mathbf{g} - \mathbf{d} \times \mathbf{h}) \end{matrix} \Bigg)$
$+(1 \times 3)$	$\Bigg(\begin{matrix} 0 \\ -(a_\mu f^\mu)\, \omega \end{matrix}$	$\begin{matrix} 0 \\ 0 \end{matrix}$	$\begin{matrix} \mathbf{a} \times \mathbf{f} \vee \sigma^4 \\ \eta \vee \left(-a^4 \mathbf{f} + f^4 \mathbf{a} \right) \end{matrix} \Bigg)$
$+(3 \times 1)$	$\Bigg(\begin{matrix} 0 \\ (b_\mu e^\mu)\, \omega \end{matrix}$	$\begin{matrix} 0 \\ 0 \end{matrix}$	$\begin{matrix} -\mathbf{b} \times \mathbf{e} \vee \sigma^4 \\ \eta \vee \left(-e^4 \mathbf{b} + b^4 \mathbf{e} \right) \end{matrix} \Bigg)$
$+(2 \times 3)$	$\Bigg(\begin{matrix} 0 \\ 0 \end{matrix}$	$\begin{matrix} \mathbf{c} \times \mathbf{f} + f^4 \mathbf{d} + (\mathbf{d} \cdot \mathbf{f}) \sigma^4 \\ \omega \vee \left(-\mathbf{d} \times \mathbf{f} + f^4 \mathbf{c} + (\mathbf{c} \cdot \mathbf{f}) \sigma^4 \right) \end{matrix}$	$\begin{matrix} 0 \\ 0 \end{matrix} \Bigg)$
$+(3 \times 2)$	$\Bigg(\begin{matrix} 0 \\ 0 \end{matrix}$	$\begin{matrix} -\mathbf{b} \times \mathbf{g} + b^4 \mathbf{h} + (\mathbf{b} \cdot \mathbf{h}) \sigma^4 \\ \omega \vee \left(-\mathbf{b} \times \mathbf{h} - b^4 \mathbf{g} - (\mathbf{b} \cdot \mathbf{g}) \sigma^4 \right) \end{matrix}$	$\begin{matrix} 0 \\ 0 \end{matrix} \Bigg)$
$+(3 \times 3)$	$\Bigg(\begin{matrix} (b_\mu f^\mu) \\ 0 \end{matrix}$	$\begin{matrix} 0 \\ 0 \end{matrix}$	$\begin{matrix} (f^4 \mathbf{b} - b^4 \mathbf{f}) \vee \sigma^4 \\ -\eta \vee (\mathbf{b} \times \mathbf{f}) \end{matrix} \Bigg)$

four-dimensional space, the basis forms can be identified with the representations of the permutation group in four indices $S(4)$. Recalling the Young Tableaux of $S(4)$, which are for this purpose effectively the same as those of $SU(4)$ [37], we have the identification.

$$dx^\mu \leftrightarrow \begin{array}{|c|}\hline \mu \\ \hline \end{array} \tag{4a}$$

$$dx^\mu \wedge dx^\nu \leftrightarrow \begin{array}{|c|}\hline \mu \\ \hline \nu \\ \hline \end{array} \tag{4b}$$

$$dx^\mu \wedge dx^\nu \wedge dx^\lambda \leftrightarrow \begin{array}{|c|}\hline \mu \\ \hline \nu \\ \hline \lambda \\ \hline \end{array} \tag{4c}$$

$$dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4 \leftrightarrow \begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline 4\\\hline\end{array} \tag{4d}$$

This pictorial representation immediately gives all the useful properties of the basis forms in the Minkowski space. First, we know the number of distinct basis forms of rank p, since the indices μ, ν, λ vary only from 1 to 4. There are $(4/p) = 4!/p!(4-p)!$ distinct (up to a permutation of indices) basis forms of rank p, where $p = 1, 2, 3$, or 4. The diagrams corresponding to these 15 basis forms are listed below (along with (4d)).

$$\begin{gathered}
\boxed{\mu} = \boxed{1} + \boxed{2} + \boxed{3} + \boxed{4} \\
\begin{array}{|c|}\hline \mu\\\hline \nu\\\hline\end{array} = \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} + \begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array} + \begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array} + \begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array} + \begin{array}{|c|}\hline 2\\\hline 4\\\hline\end{array} + \begin{array}{|c|}\hline 3\\\hline 4\\\hline\end{array} \\
\begin{array}{|c|}\hline \mu\\\hline \nu\\\hline \lambda\\\hline\end{array} = \begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array} + \begin{array}{|c|}\hline 1\\\hline 2\\\hline 4\\\hline\end{array} + \begin{array}{|c|}\hline 1\\\hline 3\\\hline 4\\\hline\end{array} + \begin{array}{|c|}\hline 2\\\hline 3\\\hline 4\\\hline\end{array}
\end{gathered} \tag{5}$$

The existence of only one four-form (diagram (4d)) follows from the fact that there is only one entirely antisymmetric combination of four indices. From rule (1d, e), we deduce that any exterior product containing four basis one-forms in four dimensions is either zero (if any two indices are equal), or is equal to the four-form (4d) up to a sign. Because of its importance, and also because of the geometry, we call this four-form the "volume element in four dimensions," and denote it by $\omega = dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4$. The permutation symmetry can then be expressed in terms of the Levi–Civita antisymmetric index symbol $\epsilon^{\mu\nu\lambda\rho}$ to give the rule

$$dx^\mu \wedge dx^\nu \wedge dx^\lambda \wedge dx^\rho = \epsilon^{\mu\nu\lambda\rho}\omega. \tag{6}$$

It is clear that in four dimensions there can be no forms of rank greater than four. It is also useful to call the unit scalar 1 the "zero-form," so that we can consider all the basis forms together with the unit as an algebraic basis of $2^4 = 16$ elements.

We now proceed with the geometrical interpretation of the basis forms. And that is the following: a basis p-form represents an element of a p-dimensional hypersurface defined by the form indices. For example, the

dx^μ have already been identified with the basis vectors of the Minkowski space. The objects $dx^\mu \wedge dx^\nu$ are area elements in the plane defined by the x^μ and x^ν coordinates. The spatial planes are those with space indices $i, j = 1, 2, 3$, which will always be denoted by Latin superscripts as $dx^i \wedge dx^j$; there are three more planes which determine the three Lorentz boosts and have elements $dx^i \wedge dx^4$, $i = 1, 2, 3$. Similarly, there are four three-dimensional hypersurfaces which are characterized by each basis three-form, of which one is the volume of ordinary space. Because of its importance, we single out the volume element of three-space, and denote it by $\eta = dx^1 \wedge dx^2 \wedge dx^3$.

One can visualize geometrically the role of the exterior product in constructing areas from lines, etc. Also the antisymmetry is just a consequence of the orientation of the geometrical elements—if one switches x^1 and x^2 in $dx^1 \wedge dx^2$ for example, one switches the x^1 and x^2 axes, with the result that the orientation of the area element is now opposite to what it was originally. In order to preserve the geometry under such redefinitions, we must accompany this switch with a change in sign, which gives $dx^1 \wedge dx^2 = -dx^2 \wedge dx^1$.

At this point, it is useful to adopt a more convenient notation, which will be used for the remainder of the paper. Henceforth, we shall denote the differentials as $\sigma^\mu = dx^\mu$. In this notation, we display all the basis forms in Minkowski spacetime, here grouped according to rank.

$$\begin{gathered}\sigma^1, \sigma^2, \sigma^3, \sigma^4,\\ \sigma^1 \wedge \sigma^2, \sigma^1 \wedge \sigma^3, \sigma^2 \wedge \sigma^3, \sigma^1 \wedge \sigma^4, \sigma^2 \wedge \sigma^4, \sigma^3 \wedge \sigma^4,\\ \sigma^1 \wedge \sigma^2 \wedge \sigma^3 = \eta, \sigma^1 \wedge \sigma^2 \wedge \sigma^4, \sigma^1 \wedge \sigma^3 \wedge \sigma^4, \sigma^2 \wedge \sigma^3 \wedge \sigma^4,\\ \sigma^1 \wedge \sigma^2 \wedge \sigma^3 \wedge \sigma^4 = \omega.\end{gathered} \tag{7}$$

As soon as one looks at the geometric objects in a space, one sees the fundamental role of the duality concept. In ordinary three-dimensional space this is well known as the association of a unit vector normal to each area element. In the case of the four-dimensional Minkowski space, the duality associates a p-form with another $(4 - p)$-form. Hence, the one-forms are dual to the three-forms, and the two-forms are dual to other two-forms. In order to describe this in a more formal manner, we must introduce the Minkowski metric as the inner product between the basis one-forms.

$$g^{\mu\nu} = (\sigma^\mu, \sigma^\nu) = 0 \qquad \mu \neq \nu, \qquad g^{\mu\mu} = (\sigma^\mu, \sigma^\mu) = (-1, -1, -1, +1). \tag{8}$$

The four-dual of any basis p-form $\sigma^{\mu_1} \wedge \cdots \wedge \sigma^{\mu_p}$ can now be formally defined and is denoted by a star (the subscript refers to the dimension of the space)

$$\underset{4}{*}\left(\sigma^{\mu_1} \wedge \cdots \wedge \sigma^{\mu_p}\right) = (-1)^{\pi_n} g^{\mu_{p+1}\mu_{p+1}} \cdots g^{\mu_n\mu_n} \sigma^{\mu_{p+1}} \wedge \cdots \wedge \sigma^{\mu_n}, \tag{9}$$

where $(-1)^{\pi_n}$ is the signature of the permutation

$$\pi_n = \begin{pmatrix} \mu_1 & \cdots & \mu_p & \cdots & \mu_n \\ 1 & \cdots & p & \cdots & n \end{pmatrix}. \tag{10}$$

Stated simply, the dual to a basis form is constructed as the exterior product of all the basis one-forms that do not appear in the original form—up to a sign. This sign is then determined by the order of the indices and the metric. We note that in actual practice, the duals will be treated in an extremely simple manner using the "vee" product defined below. The duality is subsequently discussed in greater detail in Section 4. This completes the construction of the differential forms which will be used as a basis for the Clifford algebra in Minkowski spacetime.

3. The Vee Product

Following the discussion of the previous section, we see that the exterior product is not so much an algebraic tool as it is a geometrical one—it is the means by which the elements of the geometry are built up from the differential basis. What we would like at this point is to define an algebraic product between the basis forms (7) so as to obtain a multiplicative ring structure, i.e., a closed algebra in the usual sense, with inverses, etc. This can be accomplished by defining a multiplication between the basis forms, denoted by $\vee$ "vee", which has the following properties:

$$s \vee \sigma^\mu = s\sigma^\mu \qquad s = \text{scalar, ordinary multiplication,} \tag{11a}$$

$$\sigma^\mu \vee \sigma^\nu = g^{\mu\nu} + \sigma^\mu \wedge \sigma^\nu \qquad \text{bilinear product,} \tag{11b}$$

$$\sigma^\mu \vee \sigma^\nu \vee \sigma^\lambda = \sigma^\mu \wedge \sigma^\nu \wedge \sigma^\lambda + g^{\mu\nu}\sigma^\lambda + g^{\nu\lambda}\sigma^\mu - g^{\mu\lambda}\sigma^\nu \qquad \text{triple product,} \tag{11c}$$

$$\left(\sigma^\mu + \sigma^\nu\right) \vee \sigma^\lambda = \sigma^\mu \vee \sigma^\lambda + \sigma^\nu \vee \sigma^\lambda \qquad \text{distributivity,} \tag{11d}$$

$$\left.\begin{aligned}\left(\sigma^\mu \vee \sigma^\nu\right) \vee \sigma^\lambda &= \sigma^\mu \vee \left(\sigma^\nu \vee \sigma^\lambda\right) \\ &= \sigma^\mu \vee \sigma^\nu \vee \sigma^\lambda.\end{aligned}\right\} \qquad \text{associativity.} \tag{11e}$$

The vee product therefore combines the exterior product with contractions. The vee product of any number of basis one-forms is formally defined in exactly the same way as the Wick expansion of a normal-ordered product [38]. The vee product of r one-forms is given by

DEFINITION 1.

$$\sigma^{\mu_1} \vee \cdots \vee \sigma^{\mu_r} = \sum_{k=0}^{n} \sum_{\Pi_r} (-1)^{\Pi_r} g^{\lambda_1\lambda_2} \cdots g^{\lambda_{2k-1}\lambda_{2k}} \sigma^{\lambda_{2k+1}} \wedge \cdots \wedge \sigma^{\lambda_r}, \tag{12}$$

where the permutation Π_r is defined as

$$\Pi_r = \begin{pmatrix} \lambda_1 \cdots \lambda_r \\ \mu_1 \cdots \mu_r \end{pmatrix}. \tag{13}$$

This product is given as a summation over different rank forms and permutations. To see how Definition (12) works in practice, consider the case $r = 3$. The form sum is then given by:

$$\sigma^{\mu_1} \vee \sigma^{\mu_2} \vee \sigma^{\mu_3} = \sum_{\Pi_3} (-1)^{\Pi_3} \sigma^{\lambda_1} \wedge \sigma^{\lambda_2} \wedge \sigma^{\lambda_3} + \sum_{\Pi_3} (-1)^{\Pi_3} g^{\lambda_1\lambda_2} \sigma^{\lambda_3}. \tag{14}$$

The first term is just $\sigma^{\mu_1} \wedge \sigma^{\mu_2} \wedge \sigma^{\mu_3}$. (Any permutation of this three-form merely changes its sign; we tacitly refrain from summing identical copies of a p-form, otherwise we have to include a permutation factor of $1/p!$.) The second term in the form sum is a sum of distinct one-forms with contractions; the indices and signs are given by (13) with ordering as $+g^{\mu_1\mu_2}\sigma^{\mu_3} - g^{\mu_1\mu_3}\sigma^{\mu_2} + g^{\mu_2\mu_3}\sigma^{\mu_1}$. Again, we do not sum over all possible permutations of the indices, since that would give zero from the symmetry of g. We see therefore that definition (12) in the case $r = 3$ gives (11c).

The algebraic structure is completed by giving a prescription for the vee product between basis forms of any rank. The vee product of a basis r-form with a basis s-form is given as follows [22]:

DEFINITION 2.

$$\begin{aligned} &(\sigma^{\mu_1} \wedge \cdots \wedge \sigma^{\mu_r}) \vee (\sigma^{\nu_1} \wedge \cdots \wedge \sigma^{\nu_s}) \\ &\quad = \sum_{k=0}^{\min(r,s)} (-1)^{k(r-k)} \sum_{\Pi_r} \sum_{\Pi_s} (-1)^{\Pi_r} (-1)^{\Pi_s} g^{\lambda_1\rho_1} \cdots g^{\lambda_k\rho_k} \\ &\quad\quad \cdot \sigma^{\lambda_{k+1}} \wedge \cdots \wedge \sigma^{\lambda_r} \wedge \sigma^{\rho_{k+1}} \wedge \cdots \wedge \sigma^{\rho_s}, \end{aligned} \tag{15}$$

where the permutation Π_r was given in (13) and Π_s is similarly given as:

$$\Pi_s = \begin{pmatrix} \rho_1 \cdots \rho_s \\ \nu_1 \cdots \nu_s \end{pmatrix}. \tag{16}$$

Equation (15) expresses a sum of permutations of forms; the form sum runs from $k = 0$ to $k = r$ or s, whichever is smaller. Normally, one would order the form indices, in which case the permutation factors in the formal definition which prevent multiple counting are not necessary [21, 22].

Computations of vee products are in actual practice very simple. For example, definition 2 (Eq. (15)) can be directly applied to yield the vee product between a basis two-form and a basis one-form as:

$$(\sigma^{\mu} \wedge \sigma^{\nu}) \vee \sigma^{\lambda} = \sigma^{\mu} \wedge \sigma^{\nu} \wedge \sigma^{\lambda} + g^{\nu\lambda}\sigma^{\mu} - g^{\mu\lambda}\sigma^{\nu}. \tag{17}$$

A comparison of (17) with the triple product (11c) prompts the following very useful prescription: "$(\sigma^{\mu} \wedge \sigma^{\nu}) \vee \sigma^{\lambda}$ is obtained from $\sigma^{\mu} \vee \sigma^{\nu} \vee \sigma^{\lambda}$ by deleting all terms with $g^{\mu\nu}$ in the expression for the latter." This is a rule that may be followed to obtain *any* vee product between forms, and not just for the example given.

Using these results, it is easy to establish the manipulatory consistency of the vee product. More precisely, one can show that vee products can be bracketed in any convenient way, such as

$$\begin{aligned} \sigma^{\mu} \vee \sigma^{\nu} \vee \sigma^{\lambda} \vee \sigma^{\rho} &= (\sigma^{\mu} \vee \sigma^{\nu}) \vee (\sigma^{\lambda} \vee \sigma^{\rho}) \\ &= \left((\sigma^{\mu} \vee \sigma^{\nu}) \vee \sigma^{\lambda}\right) \vee \sigma^{\rho}, \qquad \text{etc.,} \end{aligned} \tag{18a}$$

$$\left((\sigma^{\mu} \wedge \sigma^{\nu}) \vee (\sigma^{\lambda} \wedge \sigma^{\rho})\right) \vee \sigma^{\sigma} = (\sigma^{\mu} \wedge \sigma^{\nu}) \vee \left((\sigma^{\lambda} \wedge \sigma^{\rho}) \vee \sigma^{\sigma}\right). \tag{18b}$$

Even though (18a) follows directly from the associativity (11e), it is important to note that any expressions such as (18b) are valid for practical calculations. These remarks should suffice to show that one is allowed to manipulate with the $\vee$ symbol as an ordinary multiplication which is associative and distributive.

4. Theorems for the Vee Product in Minkowski Spacetime

If the general expression for the vee multiplication (12) is specialized to the case $n = 4$, with the metric $g^{\mu\mu} = (-1, -1, -1, +1)$, the result becomes quite simple. For a product of an even number of one forms we have:

Theorem 1.

$$\begin{aligned} \sigma^{\mu_1} \vee \sigma^{\mu_2} \vee \cdots \vee \sigma^{\mu_r} &= \sum_{\Pi_r} (-1)^{\Pi_r} g^{\lambda_1\lambda_2} \cdots g^{\lambda_{r-1}\lambda_r} \\ r = even \qquad &+ \sum_{\Pi_r} (-1)^{\Pi_r} \sigma^{\lambda_1} \wedge \sigma^{\lambda_2} g^{\lambda_3\lambda_4} \cdots g^{\lambda_{r-1}\lambda_r} \\ &+ \sum_{\Pi_r} (-1)^{\Pi_r} \omega \epsilon^{\lambda_1\lambda_2\lambda_3\lambda_4} g^{\lambda_5\lambda_6} \cdots g^{\lambda_{r-1}\lambda_r}. \end{aligned} \tag{19}$$

Here, ω is the volume element in spacetime, and $\epsilon^{\lambda_1\lambda_2\lambda_3\lambda_4}$ is the entirely antisymmetric Levi–Civita symbol in four spacetime indices. For r = odd, a product of an odd number of one-forms then results:

$$\begin{aligned}\sigma^{\mu_1} \vee \cdots \vee \sigma^{\mu_r} &= \sum(-1)^{\Pi_r}\sigma^{\lambda_1}g^{\lambda_2\lambda_3}\cdots g^{\lambda_{r-1}\lambda_r},\\ r = \text{odd} \qquad &+\sum(-1)^{\Pi_r}\sigma^{\lambda_1}\wedge\sigma^{\lambda_2}\wedge\sigma^{\lambda_3}g^{\lambda_4\lambda_5}\cdots g^{\lambda_{r-1}\lambda_r}.\end{aligned} \tag{20}$$

Hence, for the vee product of an *even* number of one-forms, the result is at most a sum of scalars, two-forms, four-forms. For an *odd* number of one-forms, the result is at its most complicated a sum of one-forms and three-forms.

The relationship between this algebra and the Dirac algebra has been discussed in [20, 21, 23]. We have previously used these identities (19), (20) to give a useful set of trace identities for the Dirac gamma matrices [20].

Actual computations with the vee product are facilitated enormously by the following observations: since $\sigma^\mu \vee \sigma^\nu = \sigma^\mu \wedge \sigma^\nu + g^{\mu\nu}$ (11b), it follows that

$$\sigma^\mu \vee \sigma^\nu = -\sigma^\nu \vee \sigma^\mu, \qquad \mu \neq \nu, \tag{21a}$$

$$\sigma^\mu \vee \sigma^\mu = g^{\mu\mu} \qquad (\text{no sum}). \tag{21b}$$

Thus for $\mu \neq \nu$, terms in the vee product anticommute; for $\mu = \nu$ they commute. The observation (21a) allows the basis (7) to be written directly in terms of the vee product. Because of the possibility of introducing brackets in the product and the relations (21), all reductions become trivial. For example:

$$\eta = \sigma^1 \vee \sigma^2 \vee \sigma^3, \qquad \omega = \sigma^1 \vee \sigma^2 \vee \sigma^3 \vee \sigma^4 = \eta \vee \sigma^4. \tag{22a, b}$$

Therefore:

$$\eta \vee \eta = \sigma^1 \vee \sigma^2 \vee \sigma^3 \vee \sigma^1 \vee \sigma^2 \vee \sigma^3 = -g^{11}g^{22}g^{33} = 1, \tag{23}$$

where we have anticommuted the σ's prior to contraction to obtain the minus sign. Also, from (22b),

$$\omega \vee \omega = g^{11}g^{22}g^{33}g^{44} = -1. \tag{24}$$

The calculation of the vee product is now straightforward; some of the results are reminiscent of gamma-matrix manipulations, while others appear novel.

$$\eta \vee \omega = \eta \vee \eta \vee \sigma^4 = \sigma^4, \tag{25}$$

$$\omega \vee \sigma^\mu = -\sigma^\mu \vee \omega. \tag{26}$$

To show (26), substitute $\omega \vee \sigma^\mu = \sigma^1 \vee \sigma^2 \vee \sigma^3 \vee \sigma^4 \vee \sigma^\mu$; pushing σ^μ to the left requires four exchanges of which three are negative (for unequal indices) while one is positive. Thus there is a net change of sign, yielding (26). Similar to (26) there is:

$$\sigma^i \vee \eta = \eta \vee \sigma^i, \qquad \sigma^4 \vee \eta = -\eta \vee \sigma^4. \tag{27a, b}$$

Note that (26) and (27) are independent of the scalar products g. However,

$$(\sigma^\mu \wedge \sigma^\nu) \vee (\sigma^\mu \wedge \sigma^\nu) = -g^{\mu\mu} g^{\nu\nu} \qquad \text{(metric dependent)}. \tag{28}$$

These results indicate that some of the algebraic relations are independent of the metric g, while others explicitly contain it.

We can now discuss the group property of the vee product, which is the association of a *single* basis form to every pair of individual basis forms. For example, (17) gives for two specific choices of indices,

$$(\sigma^1 \wedge \sigma^2) \vee \sigma^2 = g^{22}\sigma^1 = -\sigma^1, \tag{29a}$$

$$(\sigma^1 \wedge \sigma^2) \vee \sigma^3 = \sigma^1 \wedge \sigma^2 \wedge \sigma^3 = \eta. \tag{29b}$$

It is clear that in this manner all the basis forms (7) close under the vee product. They in fact define a finite group, and a multiplicative ring, which have as elements the basis forms (7). The group axioms are all satisfied: the unit is the scalar 1; the inverse of each form is just the form itself up to a sign, as can be verified by direct vee multiplication. For example, from (21b), (23), (24), and (28) we obtain the vee-inverses of the basis forms as:

$$(\sigma^\mu)^{-1} = \frac{1}{g^{\mu\mu}}\sigma^\mu \qquad \text{(no sum)}, \tag{30a}$$

$$(\sigma^\mu \wedge \sigma^\nu)^{-1} = \frac{-1}{g^{\mu\mu} g^{\nu\nu}}\sigma^\mu \wedge \sigma^\nu \qquad \text{(no sum)}, \tag{30b}$$

$$(\sigma^\mu \wedge \sigma^\nu \wedge \sigma^\lambda)^{-1} = \frac{-1}{g^{\mu\mu} g^{\nu\nu} g^{\lambda\lambda}}\sigma^\mu \wedge \sigma^\nu \wedge \sigma^\lambda, \qquad \omega^{-1} = -\omega. \tag{30c, d}$$

The vee group properties of the algebra of forms in Minkowski space-time can be usefully summarized in the multiplication table of all the basis forms (Table 1). In Table 1, we have employed the dual notation discussed in Section 5 to label the space two-forms as $*\sigma^i = *_3\sigma^i$ and the three-forms as $\overline{\sigma^i} = *_4\sigma^i$. It is instructive to note that a similar table for the *exterior* product would contain predominantly zero entries.

Table 1 explicitly shows that the space forms by themselves define a proper subgroup whose multiplication table is the upper left quadrant of

Table 1. This block is identical to Table 1 of Ref. [21]. For a detailed discussion of the vee-group structure, see [21, 23].

An important consequence of the vee-group structure is the behavior of the volume elements η and ω. The manipulations in this section establish their algebraic properties as follows:

THEOREM 2. *Algebraic properties of the Volume elements.*

"(*a*) *The volume element* η *commutes with all space one-forms* σ^i ($i = 1, 2, 3$), *anticommutes with* σ^4 *and* ω, *and has square equal to* $+1$.

(*b*) *The volume element* ω *anticommutes with all spacetime one-forms* σ^μ ($\mu = 1, 2, 3, 4$) *and has square equal to* -1."

Following the discussion of duals in the following section, we will show how the manipulations of forms in Minkowski spacetime rely to a great extent on the algebraic properties of the volume elements η and ω, as given by Theorem 2.

5. THE DUALITY THEOREM

The traditional definition of the dual to a form (or a tensor) is usually phrased in terms of the Levi–Civita tensor density $\epsilon^{\lambda\mu\nu\rho}$ [13, 14]. The indices may be raised and lowered using the metric $g^{\mu\nu}$, so in the Minkowski space $\epsilon_{\lambda\mu\nu\rho} = -\epsilon^{\lambda\mu\nu\rho}$. The choice made here is that $\epsilon^{1234} = +1$. The duals of the forms are given below. (Sum over lowered repeated indices; the factorials are not needed if the indices are ordered.)

$$\underset{4}{*}\, 1 = \frac{1}{4!}\epsilon_{\lambda\mu\nu\rho}\sigma^\lambda \wedge \sigma^\mu \wedge \sigma^\nu \wedge \sigma^\rho, \tag{31a}$$

$$\underset{4}{*}\, \sigma^\lambda = \frac{1}{3!}\epsilon^\lambda_{\ \mu\nu\rho}\sigma^\mu \wedge \sigma^\nu \wedge \sigma^\rho, \tag{31b}$$

$$\underset{4}{*}\, (\sigma^\lambda \wedge \sigma^\mu) = \frac{1}{2!}\epsilon^{\lambda\mu}{}_{\nu\rho}\sigma^\nu \wedge \sigma^\rho, \tag{31c}$$

$$\underset{4}{*}\, (\sigma^\lambda \wedge \sigma^\mu \wedge \sigma^\nu) = \epsilon^{\lambda\mu\nu}{}_{\rho}\sigma^\rho, \tag{31d}$$

$$\underset{4}{*}\, \omega = 1. \tag{31e}$$

We have explicitly noted the dimension four of the Minkowski space in the duals, equal to the number of indices on the Levi–Civita tensor. We will also be discussing duals in *three* dimensions; this notation prevents any confusion. It is not difficult to check that the duals defined by (31) with the sign conventions adopted are identical with those defined by (9). The formulae given here apply to the Minkowski space with metric $g^{\mu\mu} \equiv (-1, -1, -1, +1)$.

It is of interest to consider duals in the strictly three-dimensional pseudo-Euclidean subspace $g^{ii} = (-1, -1, -1)$ where now instead of (31) the three-dimensional duals are given. The definitions (9) remain, of course, the same—just the space part of the metric must be used (i, j, $k = 1, 2, 3$).

$$\underset{3}{*}\, 1 = \frac{1}{3!}\epsilon_{ijk}\sigma^i \wedge \sigma^j \wedge \sigma^k = -\eta, \tag{32a}$$

$$\underset{3}{*}\, \sigma^i = \frac{1}{2!}\epsilon^i_{jk}\sigma^j \wedge \sigma^k, \tag{32b}$$

$$\underset{3}{*}\, (\sigma^i \wedge \sigma^j) = \epsilon^{ij}_k \sigma^k, \tag{32c}$$

$$\underset{3}{*}\, \eta = 1. \tag{32d}$$

The notation $\underset{3}{*}\, \sigma^i$ stresses that this dual is taken in the three-dimensional pseudo-Euclidean subspace.

The above definitions enable one to label two forms and three forms as three- and four-duals of one-forms, respectively. This results in considerable economy of notation, which has already been employed in Table 1.

$$\begin{aligned} \sigma^2 \wedge \sigma^3 &= \underset{3}{*}\, \sigma^1, & \sigma^2 \wedge \sigma^3 \wedge \sigma^4 &= \underset{4}{*}\, \sigma^1, \\ \sigma^3 \wedge \sigma^1 &= \underset{3}{*}\, \sigma^2, & \sigma^3 \wedge \sigma^1 \wedge \sigma^4 &= \underset{4}{*}\, \sigma^2, \\ \sigma^1 \wedge \sigma^2 &= \underset{3}{*}\, \sigma^3, & \sigma^1 \wedge \sigma^2 \wedge \sigma^4 &= \underset{4}{*}\, \sigma^3. \end{aligned} \tag{33a, b}$$

We have given the above definitions in preparation for the main result of this section, the "Duality Theorem," which ties the duality operation to the vee product in an intrinsic manner. This is the following:

THEOREM 3. *"The duals are expressed algebraically as the vee product with the volume elements."—Duality Theorem*

This is true for both the three- and four-dimensional duals. Specifically, the results obtained for the four-dimensional duals are:

$$\underset{4}{*}\, 1 = -\omega, \tag{34a}$$

$$\underset{4}{*}\, \sigma^\mu = \omega \vee \sigma^\mu, \tag{34b}$$

$$\underset{4}{*}\, (\sigma^\mu \wedge \sigma^\nu) = \omega \vee (\sigma^\mu \wedge \sigma^\nu), \tag{34c}$$

$$\underset{4}{*}\, (\sigma^\mu \wedge \sigma^\nu \wedge \sigma^\rho) = -\omega \vee (\sigma^\mu \wedge \sigma^\nu \wedge \sigma^\rho), \tag{34d}$$

$$\underset{4}{*}\, \omega = -\omega \vee \omega = 1. \tag{34e}$$

Equations (34) *show explicitly that the dual of the basis forms apart from a sign is obtained by multiplying each form by* ω, *in the vee multiplication. The proof can be carried out most straightforwardly by direct verification. For example,* $\omega \vee \sigma^1$ *can be calculated; it is* $\sigma^2 \wedge \sigma^3 \wedge \sigma^4$, *which is the same as* $\underset{4}{*}\sigma^1$ *calculated from* (31b) *or* (9). *Since all the manipulations are elementary (but a little lengthy and tedious) all details are omitted. The results* (34) *can be summarized if* ζ_p *is a p-form*:

$$\underset{4}{*}\zeta_p = (-1)^{1+(1/2)p(p+1)}\omega \vee \zeta_p. \tag{35}$$

$\omega \vee \zeta_p$ *is a* $(4-p)$*-form. Using* (24) *one checks very easily from* (34) *that*

$$\underset{4}{*}\,\underset{4}{*}\zeta_p = (-1)^{p+1}\zeta_p \tag{36}$$

which is a well-known result [13].

Similar results emerge for the three-dimensional duals. Theorem 3 *gives*:

$$\underset{3}{*}1 = -\eta \vee 1 = -\eta, \tag{37a}$$

$$\underset{3}{*}\sigma^i = -\eta \vee \sigma^i, \tag{37b}$$

$$\underset{3}{*}(\sigma^i \wedge \sigma^j) = \eta \vee (\sigma^i \vee \sigma^j), \tag{37c}$$

$$\underset{3}{*}\eta = \eta \vee \eta = 1. \tag{37d}$$

Again the three-dimensional dual follows (apart from sign) by vee multiplication with η, *the volume element in three dimensions.*

Equations (37) *can be summarized for any p-form* ($p \leqslant 3$) *as*

$$\underset{3}{*}\zeta_p = (-1)^{(1/2)(p+1)(p+2)}\eta \vee \zeta_p \tag{38}$$

Applying (38) *twice leads to the known result for all p*,

$$\underset{3}{*}\,\underset{3}{*}\zeta_p = -\zeta_p. \tag{39}$$

It is particularly important that the 3-dimensional dual operation expressed in terms of η by (37) operates throughout with objects in the three-dimensional pseudo-Euclidean subspace. This allows an explicit separation of the three-dimensional and four-dimensional duality operations. To avoid cumbersome notation, the four-dimensional dual will be denoted by a *bar*, the three-dimensional dual by just a *star* (with no index) from this point on.

DEFINITION 3. For any form ζ, we denote the three-dual as $*\zeta = \underset{3}{*}\zeta$; and the 4-dual as $\bar{\zeta} = \underset{4}{*}\zeta$.

Since $\omega = \sigma^1 \vee \sigma^2 \vee \sigma^3 \vee \sigma^4 = \eta \vee \sigma^4$, there will be many relations between the 3- and 4-dimensional duals. A general characteristic for all forms ζ which contain just space forms, $\sigma^i \wedge \sigma^j$, etc., but no σ^4 is the following identity, obtained from Theorems 2 and 3 (in the notation of Definition 3).

$$\bar{\zeta} = (*\zeta) \vee \sigma^4. \tag{40}$$

The signs in the four-dimensional dual operations are just cancelled by those in the three-dimensional dual (as can be checked directly). Identity (40) actually suggests the bar notation; by observing its analogy to the adjoint spinor in the Dirac theory, $\bar{\psi} = \psi^\dagger \gamma^4$.

6. TENSOR TYPES AND THEIR ALGEBRA

The discussion up until this point has fixed the realization of the Clifford algebra in Minkowski spacetime by discussing the properties of the differential form basis. We now proceed to the field description proper. The algebra is 16 dimensional and can therefore be adjoined to a 16-dimensional real vector space. The fields which are elements of the algebra will, in general, be linear combinations of the 16 basis forms (7) with real scalar coefficients. Since the basis forms possess clear geometric properties characterized by their rank, these scalar coefficients can be interpreted as the components of antisymmetric tensor fields. This identification is accomplished as follows: the most general element of the algebra α is written as

$$\alpha = f_0 + \sum_{\mu} f_1^{\mu}\sigma^{\mu} + \frac{1}{2}\sum_{\mu\nu} f_2^{\mu\nu}\sigma^{\mu} \wedge \sigma^{\nu} + \frac{1}{3!}\sum_{\mu\nu\lambda} f_3^{\mu\nu\lambda}\sigma^{\mu} \wedge \sigma^{\nu} \wedge \sigma^{\lambda} + f_4^0\omega, \tag{41}$$

$\mu, \nu, \lambda = 1, \ldots, 4$; $\mu \neq \nu \neq \lambda$. We identify the 16 real coefficients $f_0, f_1^{\mu}, f_2^{\mu\nu}, f_3^{\mu\nu\lambda}, f_4^0$ with antisymmetric tensor components of rank zero, one, two, three and four in Minkowski spacetime. The subscript denotes the *rank* of the tensor field.

An analogous expression for the three-dimensional spatial subalgebra of the Minkowski space can be obtained from (41) by excluding any basis form containing σ^4. This consequently excludes the fourth components of tensors in (41) to give the most general element in the strictly three-dimensional subspace as

$$h_0 + \sum_i h_1^i \sigma^i + \frac{1}{2} \sum_{ij} h_2^{ij} \sigma^i \wedge \sigma^j + h_3^0 \eta, \qquad i, j = 1, 2, 3; i \neq j. \quad (42)$$

Each one of the terms in (41) and (42) defines a certain *type*; it is useful to deal with these types—which will be called "tensor types"—directly. The "form" nomenclature is here reserved exclusively for the basis forms (7).

DEFINITION 4. *Tensor types in four dimensions.*

f_0	tensor type 0, or scalar	(43a)
$f_1 = \sum_{\mu=1}^{4} f_1^{\mu} \sigma^{\mu}$	tensor type 1, or vector type	(43b)
$f_2 = \frac{1}{2} \sum_{\mu, \nu=1}^{4} f_2^{\mu\nu} \sigma^{\mu} \wedge \sigma^{\nu}$	tensor type 2, or bivector type	(43c)
$f_3 = \frac{1}{3!} \sum_{\mu, \nu, \lambda=1}^{4} f_3^{\mu\nu\lambda} \sigma^{\mu} \wedge \sigma^{\nu} \wedge \sigma^{\lambda}$	tensor type 3, or trivector type	(43d)
$f_4 = f_4^0 \omega = f_4^0 \sigma^1 \wedge \sigma^2 \wedge \sigma^3 \wedge \sigma^4$	tensor type 4.	(43e)

The idea of the tensor type is of course equally applicable to the three-dimensional subalgebra. To distinguish the three- and four-dimensional types, the three-dimensional tensor types will be denoted by boldface letters; in the case of a scalar, no distinction is necessary. Note that in all cases, we employ the *contravariant* tensor components of the fields.

DEFINITION 5. *Tensor types in three dimensions.*

h_0	scalar	(44a)
$\mathbf{h}_1 = \sum_{i=1}^{3} h_1^i \sigma^i$	tensor type 1, vector type	(44b)
$\mathbf{h}_2 = \frac{1}{2} \sum_{i,j=1}^{3} h_2^{ij} \sigma^i \wedge \sigma^j$	tensor type 2, bivector or pseudovector type	(44c)
$\mathbf{h}_3 = h_3^0 \eta = h_3^0 \sigma^1 \wedge \sigma^2 \wedge \sigma^3$	tensor type 3.	(44d)

Since the tensor types are linear combinations of the basis forms, both the exterior and vee multiplications are defined between the tensor types. In particular, we wish to determine in what way the vee product between the basis forms defines a product on the tensor components.

For example, consider the vee product between a tensor type one and a tensor type two. From (17) and (43b, c), we obtain:

$$\begin{aligned} f_2 \vee f_1 &= \left(\tfrac{1}{2} f_2^{\mu\nu} \sigma^\mu \wedge \sigma^\nu\right) \vee \left(f_1^\lambda \sigma^\lambda\right) = \tfrac{1}{2} f_2^{\mu\nu} f_1^\lambda \left(\sigma^\mu \wedge \sigma^\nu\right) \vee \sigma^\lambda \\ &= \tfrac{1}{2} f_2^{\mu\nu} f_1^\lambda \left(\sigma^\mu \wedge \sigma^\nu \wedge \sigma^\lambda + g^{\nu\lambda}\sigma^\mu - g^{\mu\lambda}\sigma^\nu\right) \\ &= f_2 \wedge f_1 + f_2^{\mu\nu} f_{1\nu} \sigma^\mu. \end{aligned} \tag{45a}$$

We have used the antisymmetry of the components of f_2 to combine the two contracted terms. Now from the analogous expression to (17), we can evaluate the vee product the other way:

$$\begin{aligned} f_1 \vee f_2 &= \tfrac{1}{2} f_1^\lambda f_2^{\mu\nu} \sigma^\lambda \vee \left(\sigma^\mu \wedge \sigma^\nu\right) \\ &= \tfrac{1}{2} f_1^\lambda f_2^{\mu\nu} \left(\sigma^\lambda \wedge \sigma^\mu \wedge \sigma^\nu + g^{\lambda\mu}\sigma^\nu - g^{\lambda\nu}\sigma^\mu\right) \\ &= f_1 \wedge f_2 + f_{1\mu} f_2^{\mu\nu} \sigma^\nu. \end{aligned} \tag{45b}$$

The vee product of a tensor type one with a tensor type two is therefore the sum of a tensor type one with a tensor type three. The symmetric part of the product is the tensor type three $f_1 \wedge f_2$; the antisymmetric part is the tensor type one $f_{1\mu} f_2^{\mu\nu} \sigma^\nu$. This may be surprising, since it is the opposite of what one would naively expect. To show this, one can use the symmetries of the indices and the basis forms to verify that the anticommutators and commutators are equal to:

$$\{f_1, f_2\} = f_1 \vee f_2 + f_2 \vee f_1 = 2 f_1 \wedge f_2, \tag{46a}$$

$$[f_1, f_2] = f_1 \vee f_2 - f_2 \vee f_1 = 2 f_{1\mu} f_2^{\mu\nu} \sigma^\nu. \tag{46b}$$

In actual computations, the rank indices may be dispensed with; a convenient notation that is close to physical usage is: lower case Italic letters for vector types (one), and upper case Italic letters for tensors of type two and higher. From the vee product between the basis one-forms (11b), we obtain the vee product between the vector types a and b as

$$\begin{aligned} a \vee b &= \left(a^\mu \sigma^\mu\right) \vee \left(b^\lambda \sigma^\lambda\right) = a^\mu b^\lambda \left(g^{\mu\lambda} + \sigma^\mu \wedge \sigma^\lambda\right) \\ &= (a, b) + a \wedge b, \end{aligned} \tag{47a}$$

$$(a, b) = a_\mu b^\mu = -a^1 b^1 - a^2 b^2 - a^3 b^3 + a^4 b^4. \tag{47b}$$

It follows trivially that the vee product of a vector a with itself gives the Minkowski scalar quadratic form.

$$a \vee a = (a, a) = a_\mu a^\mu. \tag{48}$$

It is important to note that the vee product of two tensor types in general does *not* consist of a contraction plus an exterior product. In this respect, the vee product is entirely distinct from related products that have been previously defined by other authors as just a contraction plus an exterior product. A simple example will illustrate this point. The vee product of two tensors of type two, F and G, is calculated from (15) and (6).

$$\begin{aligned} F \vee G &= \left(\frac{1}{2} F^{\mu\nu} \sigma^\mu \wedge \sigma^\nu\right) \vee \left(\frac{1}{2} G^{\lambda\rho} \sigma^\lambda \wedge \sigma^\rho\right) \\ &= \frac{1}{4} F^{\mu\nu} G^{\lambda\rho} (\sigma^\mu \wedge \sigma^\nu) \vee (\sigma^\lambda \wedge \sigma^\rho) \\ &= -\frac{1}{2} F_{\mu\nu} G^{\mu\nu} + F^\rho_\mu G^{\mu\lambda} \sigma^\rho \wedge \sigma^\lambda + \frac{\omega}{4} F^{\mu\nu} G^{\rho\lambda} \epsilon^{\mu\nu\rho\lambda}. \end{aligned} \tag{49}$$

Hence, the vee product of two tensors of type two is the sum of a scalar, a tensor type two, and a tensor type four. Expression (49) consists of a double contraction, a single contraction, and an entire antisymmetrization, showing how the vee product combines tensor, matrix, and dyadic products. The vee product of F with itself is physically suggestive, since, when F is the electromagnetic field [22], the two terms are the invariants of the Lorentz transformations of the field F [16], or the Lagrangian terms

$$F \vee F = -\frac{1}{2} F_{\mu\nu} F^{\mu\nu} + \frac{\omega}{4} F^{\mu\nu} F^{\rho\lambda} \epsilon^{\mu\nu\rho\lambda}. \tag{50}$$

This is one example of physically relevant quantities which appear from just the algebraic structure. The vee multiplication between all the tensor types in Minkowski spacetime is worked out in the following section. Closed form expressions of the vee product of an arbitrary number of *vector* types were given in [20], where they were used to obtain a set of generalized trace identities for the Dirac gamma matrices.

7. Applications of the Duality Theorem to the Tensor Types

In this section, the description of tensor fields in terms of tensor types is completed by giving all the vee products between the tensor types (43, 44).

Before we do this, however, we show how an application of the Duality Theorem simplifies the algebra to a tremendous extent. First we write down the duals for the tensor types (43), which, as in the case of the basis forms, are given as the vee products with ω (34).

$$\bar{f}_0 = -f_0\sigma^1 \wedge \sigma^2 \wedge \sigma^3 \wedge \sigma^4 = -\omega f_0, \tag{51a}$$

$$\bar{f}_1 = \frac{1}{3!}\left(f_1^{\mu}\epsilon^{\mu}_{\nu\lambda\rho}\right)\sigma^{\nu} \wedge \sigma^{\lambda} \wedge \sigma^{\rho} = \omega \vee f_1, \tag{51b}$$

$$\bar{f}_2 = \frac{1}{2}\left(\frac{1}{2}f_2^{\mu\nu}\epsilon^{\mu\nu}{}_{\lambda\rho}\right)\sigma^{\lambda} \wedge \sigma^{\rho} = \omega \vee f_2, \tag{51c}$$

$$\bar{f}_3 = \left(\frac{1}{3!}f_3^{\mu\nu\lambda}\epsilon^{\mu\nu\lambda}{}_{\rho}\right)\sigma^{\rho} = -\omega \vee f_3, \tag{51d}$$

$$\bar{f}_4 = f_4^0 = -\omega \vee f_4. \tag{51e}$$

The simplification afforded by the Duality theorem resides in the fact that we do not need to manipulate tensors of type two, three and four directly; it is sufficient to consider only vector types and the volume elements. Therefore, the algebra of tensor types is in reality a *vector* algebra (albeit with additional intrinsic structure).

To begin with, as in the case of the basis forms in (33b), any tensor type three f_3 can be identified with the dual of a vector type j_1 as follows, by using (51b, d).

$$f_3 = \bar{j}_1 = \omega \vee j_1; \qquad j_1 = \frac{1}{3!}\sum f_3^{\nu\lambda\rho}\epsilon^{\nu\lambda\rho}{}_{\mu}\sigma^{\mu}. \tag{52}$$

Hence, any products involving f_3 can be evaluated using the associated vector type j_1. By the same token, any tensor type four is just a scalar times the volume element ω.

$$f_4 = -\bar{j}_0 = \omega j_0; \qquad j_0 = f_4^0 \qquad \text{a scalar.} \tag{53}$$

With this identification (53), manipulations with tensors of type four are trivial.

We now proceed with the vector decomposition of the type two fields. The Duality theorem in the strictly three-dimensional subspace (37) gives the space duals of the spatial tensor types (44) as the vee-product with the volume element η.

$$*h_0 = -\eta h_0, \tag{54a}$$

$$*\mathbf{h}_1 = \tfrac{1}{2}\left(h_1^i\epsilon^i_{jk}\right)\sigma^j \wedge \sigma^k = -\eta \vee \mathbf{h}_1, \tag{54b}$$

$$*\mathbf{h}_2 = \left(\tfrac{1}{2}h_2^{ij}\epsilon^{ij}_k\right)\sigma^k = \eta \vee \mathbf{h}_2, \tag{54c}$$

$$*\mathbf{h}_3 = h_3^0 = \eta \vee \mathbf{h}_3. \tag{54d}$$

It follows that any spatial tensor type two $\mathbf{h}_2$ can be identified with the space-dual of a space vector $\mathbf{t}_1$, in a manner entirely analogous to the above discussion in four dimensions. Using (54b, c) we obtain the identity:

$$\mathbf{h}_2 = *\mathbf{t}_1; \qquad \mathbf{t}_1 = \tfrac{1}{2}\epsilon^{ijk}h_2^{ij}\sigma^k. \tag{55}$$

Now, it is clear that any tensor type two F in spacetime can be decomposed into space and spacetime components as

$$F = \tfrac{1}{2}F^{\mu\nu}\sigma^\mu \wedge \sigma^\nu = \tfrac{1}{2}F^{ij}\sigma^i \wedge \sigma^j + F^{i4}\sigma^i \wedge \sigma^4. \tag{56}$$

Using the identification (55), we identify the space part of F with the space dual of a vector type $\mathbf{d}$, and then use the Duality theorem (54b) to write this as just the vee-product of $\mathbf{d}$ with η. Next, call the components F^{i4} the components of another three-dimensional vector $\mathbf{c}$ to obtain what is known as the "canonical decomposition of tensor types."

THEOREM 4. *Canonical Decomposition.*

(i) A vector type a can be trivially decomposed into space and time components as

$$a = \mathbf{a} + a^4\sigma^4. \tag{57}$$

(ii) A tensor type two F can be decomposed into space and spacetime components as

$$F = \mathbf{c} \vee \sigma^4 + \eta \vee \mathbf{d}, \tag{58a}$$

where

$$c^i = F^{i4} \qquad \textit{and} \qquad d^i = -\tfrac{1}{2}\epsilon^{ijk}F^{jk}. \tag{58b}$$

In the case when F is the ordinary electromagnetic field, this corresponds to the familiar decomposition into the electric and magnetic field vectors [22].

The adoption of notation from vector algebra leads to a more transparent understanding of the manipulations involved, and serves to make the connection to the traditional formalisms. The scalar and vector products of three-dimensional vectors are defined as the usual combinations of the vector components,

$$(\mathbf{a} \cdot \mathbf{b}) = a^1b^1 + a^2b^2 + a^3b^3, \qquad \text{and} \qquad (\mathbf{a} \times \mathbf{b})^k = \sum a^ib^j\epsilon^{ijk}. \tag{59a, b}$$

The vector type $\mathbf{a} \times \mathbf{b}$ denotes the components (59b) expanded on the one-form basis. From (54, 55), we see that $\mathbf{a} \times \mathbf{b}$ and $\mathbf{a} \wedge \mathbf{b}$ are dual.

$$\mathbf{a} \times \mathbf{b} = \sum a^i b^j \epsilon^{ijk} \sigma^k, \qquad \mathbf{a} \wedge \mathbf{b} = *\mathbf{a} \times \mathbf{b} = -\eta \vee \mathbf{a} \times \mathbf{b}. \tag{60}$$

The canonical decomposition and the Duality Theorem allow us to write the vee product of two vector fields a, b in a conceptually clear way. From (47), (57), (58) and (59), (60), we obtain the expression for the vee product:

$$a \vee b = -(\mathbf{a} \cdot \mathbf{b}) + a^4 b^4 - \eta \vee \mathbf{a} \times \mathbf{b} + (b^4 \mathbf{a} - a^4 \mathbf{b}) \vee \sigma^4. \tag{61}$$

This product is reminiscent of the original quaternion multiplication. Compare (61) with Eq. (80) to see in what way it is a generalization of the quaternion product. The vee product of two purely spatial vectors assumes a very suggestive form. It is obtained from (61) by setting a^4 and b^4 equal to zero, and is a simple combination of the scalar and vector products (59).

$$\mathbf{a} \vee \mathbf{b} = -(\mathbf{a} \cdot \mathbf{b}) - \eta \vee \mathbf{a} \times \mathbf{b}. \tag{62}$$

These identities show clearly how the vee product generalizes the usual vector algebra. Proceeding to a discussion of type two fields, the canonical decomposition can be utilized to display the hypercomplex character of the vee multiplication in a particularly interesting manner. We canonically decompose each of the following type two fields as in (58):

$$F = \mathbf{c} \vee \sigma^4 + \eta \vee \mathbf{d}, \qquad G = \mathbf{g} \vee \sigma^4 + \eta \vee \mathbf{h}. \tag{63a, b}$$

Then, the vee product of two type two fields F and G can be written in terms of the vee products of the associated vectors $\mathbf{c}$, $\mathbf{d}$, $\mathbf{g}$ and $\mathbf{h}$, as follows:

$$F \vee G = (\mathbf{d} \vee \mathbf{h} - \mathbf{c} \vee \mathbf{g}) + \omega \vee (\mathbf{c} \vee \mathbf{h} + \mathbf{d} \vee \mathbf{g}). \tag{64}$$

The product (64) of the two fields (63) is in a form akin to the usual complex product, except that here, the place of scalars is taken by three-dimensional vectors, and the elements η, σ^4 and ω take on the role of hypercomplex bases. In this context, note that η, σ^4 and ω form a mutually anticommuting basis (Theorem 2a) (see also end of Section 8).

It is now possible to give the vee products between *all* tensor types in Minkowski spacetime, by employing the canonical decomposition and the Duality Theorem. These products are displayed in Table 2 in a notation which is explained here. A 16-component field α, which is the sum of all tensor types in four dimensions (41) is here given in the following notation:

$$\begin{aligned} \alpha &= a_0 + a + \mathbf{c} \vee \sigma^4 + \eta \vee \mathbf{d} + \omega \vee b + \omega b_0 \\ &\leftrightarrow \begin{pmatrix} a_0 & a & \mathbf{c} \vee \sigma^4 \\ b_0 \omega & \omega \vee b & \eta \vee \mathbf{d} \end{pmatrix}. \end{aligned} \tag{65}$$

The field α contains all the tensor types: the *scalar* a_0; the *vector* type a; the tensor type *two* has been canonically decomposed via (58) so that it can be written in terms of the two space vectors $\mathbf{c}$ and $\mathbf{d}$; the tensor type *three* is written as the four-dual of a vector type b as in (52); and the tensor type *four* is similarly written as the four-dual of a scalar b_0 as in (53).

Now define another 16-component field β in the notation of (65), with corresponding tensor types,

$$\beta \leftrightarrow \begin{pmatrix} e_0 & e & \mathbf{g} \vee \sigma^4 \\ f_0\omega & \omega \vee f & \eta \vee \mathbf{h} \end{pmatrix}. \tag{66}$$

The most general vee product of fields in Minkowski spacetime is $\alpha \vee \beta$. Since α and β have in general 16 components, this product will have a total number of $16 \times 16 = 256$ scalar terms. However, the distributivity of the vee product and the compact notation introduced above enable us to display this product as a sum of nine groups of terms, each of which corresponds to the vee product between two distinct tensor types. Therefore, Table 2 can be used to read off vee products between the tensor types directly.

For example, the vee product of two *vector* types (61) appears as the (1×1) entry in the table. In general, the vee product of a *type* r field with a *type* s field is given by the $(r \times s)$ entry in Table 2. We have not included the scalar and type four terms, since they are trivial. An inspection of Table 2 illustrates the fact that all manipulations of tensor types in Minkowski spacetime can be accomplished using the traditional three-dimensional vector algebra, the volume elements, and the vee product.

8. Inverses of Tensor Types and the Frobenius–Hurwitz Theorems

A novel property of the vee product is that, unlike the exterior product and the ordinary vector product, the vee product allows *division* of the tensor types. This result is contained in the following theorem:

THEOREM 5. "*Every distinct tensor type has a uniquely defined two-sided inverse in vee.*"

It is important to note that a combination of distinct tensor types may not in general possess an inverse, hence the property discussed here is one of "sectional divisibility" [21]. The existence of an inverse of all combinations of tensor types, i.e., a 16-component field (41), is forbidden by the Frobenius Theorem on division algebras. (See [39, 40].)

The simplest example is the inverse of a vector field a. It is trivial to check from the vee product (48) that the inverse of the vector (type one) field a is just

$$a^{-1} = \frac{a}{(a, a)} = \frac{a}{a_\mu a^\mu}. \tag{67}$$

We note that a singularity arises in the case of a null vector $(a, a) = 0$. This is a characteristic of the noncompactness of the Minkowski manifold (and not of the vee product). Hence, the vee inverse of vectors is not defined on the light cone.

In the case of a tensor type two the inverse is not quite so simple, but reveals some very interesting physical properties. The inverse of a tensor type two $F = \frac{1}{2}F^{\mu\nu}\sigma^\mu \wedge \sigma^\nu$ is given by:

$$F^{-1} = \frac{\left(\lambda F^{k4} + \frac{1}{2}\mu F^{ij}\epsilon^{ijk}\right)\sigma^k \wedge \sigma^4 + \left(\frac{1}{2}\lambda F^{ij}\epsilon^{ijk} - \mu F^{k4}\right) * \sigma^k}{\lambda^2 + \mu^2}, \tag{68a}$$

where the scalars λ and μ are the combinations of the tensor components

$$\lambda = -\tfrac{1}{2}F_{\mu\nu}F^{\mu\nu}, \qquad \mu = +\tfrac{1}{4}F^{\mu\nu}F^{\lambda\rho}\epsilon^{\mu\nu\lambda\rho}. \tag{68b}$$

It is easy to check that this is indeed a two-sided inverse by using (50).

We see that in the act of taking the inverse of F the two invariant quantities λ and μ are produced rather naturally. When F is identified with the physical electromagnetic field, then λ and μ are the invariants of the Lorentz transformations, and the combination $(\lambda)^2 + (\mu)^2$ is the invariant of the duality rotations of the electromagnetic field [16]. Since we have not yet discussed transformations of the tensor types, this is an example of physically relevant quantities (invariants) which appear from just the algebraic properties of the tensor types. It will be shown later that taking the inverse is equivalent to a specific duality rotation, plus a dilatation (in a separate communication).

An application of the canonical decomposition of the type two field $F = \mathbf{c} \vee \sigma^4 + \eta \vee \mathbf{d}$ (58) simplifies the expression for the inverse F^{-1} and helps clarify the analogy with the electromagnetic field. The inverse of the type two field F is given by:

$$F^{-1} = \frac{(\lambda\mathbf{c} - \mu\mathbf{d}) \vee \sigma^4 + \eta \vee (\lambda\mathbf{d} + \mu\mathbf{c})}{\lambda^2 + \mu^2}, \tag{69a}$$

where the scalars λ and μ are defined by the expressions

$$\lambda = |\mathbf{c}|^2 - |\mathbf{d}|^2, \qquad \mu = -2(\mathbf{c} \cdot \mathbf{d}). \tag{69b}$$

Elsewhere [21, 22], we have discussed the physical electromagnetic field $F = \mathbf{E} \vee \sigma^4 + \eta \vee \mathbf{B}$, with $E^i = F^{i4}$, $B^i = -\frac{1}{2}\sum \epsilon^{ijk} F^{jk}$. If we write the inverse of this field as a new electromagnetic field $F^{-1} = F' = \mathbf{E}' \vee \sigma^4 + \eta \vee \mathbf{B}'$, then from (58, 69), the new electric and magnetic fields assume the form

$$\mathbf{E}' = \frac{\lambda \mathbf{E} + \nu \mathbf{B}}{\lambda^2 + \nu^2}, \qquad \mathbf{B}' = \frac{\lambda \mathbf{B} - \nu \mathbf{E}}{\lambda^2 + \nu^2}. \tag{70}$$

It is easy to check that the inverse fields have the correct units.

The Lorentz and duality rotation invariants are respectively given by the following (with $\nu = -\mu$),

$$\lambda = |\mathbf{E}|^2 - |\mathbf{B}|^2, \qquad \nu = +2(\mathbf{E} \cdot \mathbf{B}), \tag{71a}$$

$$(\lambda)^2 + (\nu)^2 = |\mathbf{E}|^2 + |\mathbf{B}|^2 - 4|\mathbf{E} \times \mathbf{B}|^2. \tag{71b}$$

The inverses of the tensor types three and four are obtained by using the identities (52), (53) to express them as duals of tensors of type one and zero, respectively. Then, the inverses are simply obtained,

$$f_3^{-1} = (\bar{j})^{-1} = \frac{\bar{j}}{(j, j)} = \frac{\omega \vee j}{j_\mu j^\mu}, \tag{72}$$

$$f_4^{-1} = (\bar{j}_0)^{-1} = \frac{-\bar{j}_0}{(j_0)^2} = \frac{\omega}{j_0}. \tag{73}$$

If one wants the inverse of the tensor type three in terms of the original f_3 components, then one can substitute (52) into (72) to obtain the inverse of a tensor type three after a simple calculation. (Compare (30c).)

$$f_3^{-1} = \frac{-f_3}{\frac{1}{3!} f_{3\mu\nu\lambda} f_3^{\mu\nu\lambda}}. \tag{74}$$

The denominator is the scalar (invariant) combination of the antisymmetric tensor components. The inverses are easily verified by direct computation.

We should mention at this point the fact that the algebra considered here does not have a norm. This is in keeping with the Hurwitz theorem [39, 40]. The nearest thing to a norm in this algebra is the scalar part of the vee product of a field with itself, denoted by the operator S.

DEFINITION 7.

$$S(\alpha) \equiv \text{scalar part of } (\alpha \vee \alpha). \tag{75}$$

If we consider the tensor types separately, then $S(f_k)$ is in each case given by

$$S(f_0) = (f_0)^2, \tag{76a}$$

$$S(f_1) = (f_1, f_1) = f_{1_\mu} f_1^\mu, \tag{76b}$$

$$S(f_2) = -\frac{1}{2} f_{2_{\mu\nu}} f_2^{\mu\nu}, \tag{76c}$$

$$S(f_3) = -\frac{1}{3!} f_{3_{\mu\nu\lambda}} f_3^{\mu\nu\lambda}, \tag{76d}$$

$$S(f_4) = -\left(f_4^0\right)^2. \tag{76e}$$

It is easy to see the resemblance of $S(\alpha)$ to the norm usually defined in division and hypercomplex algebras as $N(\alpha)$. The inverses of the tensor types zero, one, three and four can all be written in the form $\alpha^{-1} = \alpha/S(\alpha)$, which corresponds to the expression for the inverse in hypercomplex algebras $\alpha^{-1} = \tilde{\alpha}/N(\alpha)$ ($\tilde{\alpha}$ denotes the conjugate of α in the usual manner). This simple correspondence does not however hold true for the inverse of the tensor type two.

The S operator is distributive among distinct tensor types. Hence the evaluation of $S(\alpha)$, where α is now a general 16-component field (41), reduces to evaluating the S operator on the separate tensor types (76). We therefore have the expression for the S operator acting on the most general 16-component field α (41) as

$$S(\alpha) = (f_0)^2 + f_{1_\mu} f_1^\mu - \frac{1}{2} f_{2_{\mu\nu}} f_2^{\mu\nu} - \frac{1}{3!} f_{3_{\mu\nu\lambda}} f_3^{\mu\nu\lambda} - \left(f_4^0\right)^2, \tag{77a}$$

or equivalently, in terms of the field identification (65),

$$S(\alpha) = (a_0)^2 + a_\mu a^\mu + b_\mu b^\mu + c^i c^i - d^i d^i - (b_0)^2 \tag{77b}$$

The general expression for $S(\alpha)$ is a sum of 16 squares, which are not, however, positive-definite.

Not unrelated to this discussion is the fact that physical observables, such as scattering cross sections, etc., correspond to states of the S-operator (see Ref. [20, 24]).

It is worthwhile pointing out the transparent manner in which the Hurwitz theorem and its extension [39–42] can be discussed in this context. The result is expressible as the "four-square" theorem [41], and can be obtained as follows. Consider the four-component field β which is a combination of a scalar a_0 with the space-dual of a spatial vector $\mathbf{a}$ (54b),

$$\beta = a_0 + \eta \vee \mathbf{a}. \tag{78a}$$

Now define the conjugate of β, denoted by a tilde as

$$\tilde{\beta} = a_0 - \eta \vee \mathbf{a}. \tag{78b}$$

The vee product of $\tilde{\beta}$ and β is a sum of four squares, which is, moreover, positive-definite. Hence, we can define a true norm in the usual manner.

$$N(\beta) = \tilde{\beta} \vee \beta = (a_0)^2 + \mathbf{a} \cdot \mathbf{a} = (a_0)^2 + a^i a^i. \tag{79}$$

Now the vee product between $\tilde{\beta}$ and another four-component combination γ is yet another four-component field of the *same type*, as can be seen directly.

$$\gamma = b_0 + \eta \vee \mathbf{b}, \tag{80a}$$

$$\tilde{\beta} \vee \gamma = (a_0 b_0 + \mathbf{a} \cdot \mathbf{b}) + \eta \vee (a_0 \mathbf{b} - b_0 \mathbf{a} + \mathbf{a} \times \mathbf{b}). \tag{80b}$$

The four-component fields of type (78) therefore define a closed subalgebra under the vee product (with or without conjugation). It is easy to see that this subalgebra is isomorphic to the quaternion algebra, since the basis $\{\eta \vee \sigma^i\}$ or, equivalently, $\{* \sigma^i\}$, $i = 1, 2, 3$, generates the quaternion multiplication (see Table 1 and Refs. [39–42]).

A simple calculation using the vee-product rules shows that the four-component fields of type (78) satisfy the following identity for the norms (79):

$$N(\beta) N(\gamma) = N(\tilde{\beta} \vee \gamma). \tag{81}$$

This is the identity defining a "normed algebra," which expresses the product of two sums of four squares as another sum of four squares. Historically, it is known as the "four square theorem" [41]. The important point is that one *cannot* generalize this result to n squares within an associative framework. This fact can be very easily deduced from the results of this paper by considering all other combinations of tensor types such as β which have anywhere from 4 to 16 components. In most cases, one cannot define a conjugation such that $\tilde{\beta} \vee \beta$ is a scalar. In some cases where $\tilde{\beta} \vee \beta$ *is* a scalar, then $\tilde{\beta} \vee \gamma$ is not the same type as β and γ, and therefore does not define a closed subalgebra.

There are only two cases where $\tilde{\beta} \vee \beta$ is scalar and $\tilde{\beta} \vee \gamma$ is of the same type as β and γ. One is the quaternion algebra discussed above. The other is the Clifford algebra N_1 which is the same size as the quaternion algebra but its basis has squares $(+1, +1, -1)$ instead of $(-1, -1, -1)$. The algebra N_1 provides an extension of the Hurwitz theorem to the case when the normed identity (81) is satisfied even when the sum of four squares is not positive definite [42]. This result can be checked here by using the basis $\{\eta, \sigma^4, \omega\}$ for N_1. We should also mention that the algebra N_1 is related to the real two-component spinors (see Ref. [23, 42]). This and related points are discussed in detail in a comprehensive treatment of algebras with three anticommuting elements [42, 43].

The results of this section underline the fact that, even though the Clifford algebra in Minkowski spacetime is neither a normed nor a division algebra, it possesses certain useful features of both.

Conclusion

In this paper we have constructed a framework for physical tensor fields in spacetime. The key points were simplicity and ease of manipulation, a transparent geometrical interpretation, and a complete absence of tensor indices and representation matrices. The algebra is directly tied to the geometry of the Minkowski space. Following the discussion in the text detailing the logical connection to the more familiar methods of field description, this scheme appears as a natural extension of the usual vector algebra to include all antisymmetric tensors in four dimensions.

Some of the physical applications made possible by this particular field description have appeared elsewhere [20, 22, 24]. These, together with those results mentioned in this paper, should indicate the practical value of this formalism in describing physics in spacetime.

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Note added in proof. The use of differential forms to realize Clifford algebras is also discussed in [44–46]. Professor Kähler independently introduced the $\vee$ notation to denote the multiplication of differential forms which realizes the Clifford algebra.

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